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ARTICLE

Muonic–electronic quasi molecules based on a fully stripped multicharged ion

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Abstract: In our previous paper (*Can. J. Phys.* **91**, 715 (2013) doi: 10.1139/cjp-2013-0077) we studied a system consisting of a proton, a muon, and an electron; the muon and the electron being in circular states. The study was motivated by numerous applications of muonic atoms and molecules, where one of the electrons is substituted by the heavier lepton μ^-. We demonstrated that in such a μpe quasi molecule, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem — a result that may seem counterintuitive. In other words, the muon rapidly revolves in a circular orbit about the axis connecting the proton and electron while this axis slowly rotates following a relatively slow electronic motion. We showed that the spectral lines, emitted by the muon in the quasi molecule, μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom. In the present paper we generalize this study by replacing the proton in the μpe quasi molecule by a fully stripped ion of nuclear charge $Z > 1$. We show that in this case, just as in the previously studied case of $Z = 1$, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem. For this to be valid, the ratio of the muonic and electronic angular momenta should be slightly greater than in the case of $Z = 1$. We demonstrate that the binding energies of the muon for $Z > 1$ are much greater than for $Z = 1$ at any finite value of the nucleus–electron distance. Finally we show that the red shift of the spectral lines emitted by the muon (compared to the spectral lines of the corresponding muonic hydrogen-like ion of nuclear charge Z) decreases as Z increases. However, the relative red shift remains within the spectral resolution of available spectrometers at least up to $Z = 5$. Observing this red shift should be one of the ways to detect the formation of the quasi molecules, μZe.

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Résumé : Dans une publication antérieure (*Can. J. Phys.* **91**, 715 (2013) doi: 10.1139/cjp-2013-0077), nous avons étudié un système consistant en un proton, un muon et un électron, le muon et l'électron se déplaçant sur des orbites circulaires. Cette étude était motivée par plusieurs applications d'atomes et de molécules muoniques, où un des électrons est remplacé par un μ^-. Nous avons démontré que dans une telle quasi-molécule μpe, le mouvement du muon peut représenter un sous-système rapide, alors que le mouvement de l'électron peut représenter un sous-système lent — un résultat qui peut sembler contre-intuitif. Nous avons montré que les raies spectrales émises par le muon dans cette quasi-molécule μpe montrent un déplacement vers le rouge comparées aux raies spectrales qui auraient été émises par le muon dans un atome d'hydrogène muonique. Nous généralisons ici cette première étude en remplaçant le proton de la quasi-molécule μpe par ion de charge nucléaire $Z > 1$ et complètement ionisé. Nous montrons que dans ce cas, comme dans l'étude précédente avec $Z = 1$, le mouvement du muon peut représenter un sous-système rapide et celui de l'électron un sous-système lent. Pour que ceci soit valide, le rapport des moments angulaires, du muon sur l'électron, doit être légèrement plus élevé que dans le cas $Z = 1$. Nous démontrons que l'énergie de liaison du muon pour $Z > 1$ est beaucoup plus grande que pour le cas $Z = 1$, pour toute valeur finie de la distance électron–noyau. Nous montrons finalement que le déplacement vers le rouge des raies spectrales émises par le muon (comparées aux raies spectrales de l'ion muonique de type H de charge nucléaire Z) décroît lorsque Z augmente. Cependant, le déplacement vers le rouge relatif demeure à l'intérieur de la résolution spectrale des spectromètres disponibles, et ce jusqu'à $Z = 5$. L'observation de ce déplacement vers le rouge pourrait donc être une façon de détecter la formation de quasi-molécules μZe. [Traduit par la Rédaction]

1. Introduction

In our previous paper [1] we studied a system consisting of a proton, a muon, and an electron, (a μpe system or quasi molecule) the muon and the electron being in circular states. The study was motivated by numerous applications of muonic atoms and molecules, where one of the electrons is substituted by the heavier lepton μ^-. Among the applications are muon-catalyzed fusion (see, e.g., refs. 2–4 and references therein), a laser-control of nuclear processes [5], and is a search for strongly interacting massive particles (SIMPs) proposed as dark matter candidates and as candidates for the lightest supersymmetric particle (see, e.g., ref. 6 and references therein). So, in ref. 1 we demonstrated that, in a μpe quasi molecule, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem — a result that might seem counterintuitive. In other words, the muon rapidly revolves in a circular orbit about the axis connecting the proton and electron while this axis slowly rotates following a relatively slow electronic motion.

We used in ref. 1 a classical analytical description to find the energy terms of such a system (i.e., dependence of the energy of the muon on the distance between the proton and electron). We found that there is a double-degenerate energy term. We demonstrated that it corresponds to a stable motion.

Then we unfroze the slow subsystem and analysed a slow revolution of the axis connecting the proton and electron. The slow subsystem could be treated as a modified "rigid rotator" consisting of

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the electron, the proton, and the ring, over which the muon charge is uniformly distributed, all distances within the system being fixed. We derived the condition required for the validity of the separation into the rapid and slow subsystems.

Finally, in ref. 1, we showed that the spectral lines, emitted by the muon in the quasi molecule μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom (in the μp-subsystem). The relative values of this red shift, which is a "molecular" effect, are significantly greater than the resolution of available spectrometers and thus can be observed. Observing this red shift should be one of the ways to detect the formation of such muonic–electronic negative hydrogen ions.

In the present paper we generalize the study from ref. 1 by replacing the proton in the μpe quasi molecule by a fully stripped ion of a nuclear charge $Z > 1$. We denote such quasi molecules μZe. We show that in this case, just as in the previously studied case of $Z = 1$, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem. For this to be valid, the ratio of the muonic and electronic angular momenta should be slightly greater than in the case of $Z = 1$.

We demonstrate that the binding energies of the muon for $Z > 1$ are much greater than for $Z = 1$ at any finite value of the nucleus–electron distance. This is because for $Z = 1$ the total charge of the nucleus–electron subsystem is zero, so that the muon is only relatively weakly bound compared to the case of $Z > 1$, where the total charge of the nucleus–electron subsystem is positive.

We also show that the red shift of the spectral lines emitted by the muon (compared to the spectral lines of the corresponding muonic hydrogen-like ion of the nuclear charge, Z) decreases as Z increases. However, the relative red shift remains within the spectral resolution of available spectrometers at least up to $Z = 5$. Observing this red shift should be one of the ways to detect the formation of the quasi molecules μZe.

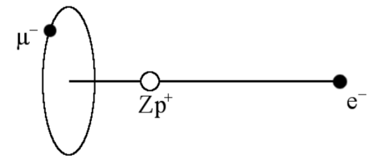
2. Analytical solution for classical energy terms of the rapid subsystem

In this study we consider a system where a muon rotates in a circle perpendicular to the axis connecting a nucleus of charge Z and an electron (see Fig. 1). Atomic units are used in this article, just as in ref. 1.

In this configuration, for the case of $Z = 1$, the muon is rotating in a circular orbit perpendicular to the proton–electron axis, while the axis rotates around the proton. It was shown in ref. 1 that for a certain range of the ratio of the muon and electron angular momenta, the muon would have a much larger revolution frequency than the electron and would therefore be considered a rapid subsystem, the electron being the slow subsystem. If the z-axis is the proton–electron axis with the origin at the proton, and the electron at $z = R$, then the equilibrium position of the muon's orbit may be anywhere on the negative half of the z-axis (i.e., "behind" the proton). The energy terms for the muon (the dependence of the energy of the muon on the distance R between the proton and electron) were obtained for $Z = 1$; the two coinciding energy terms had a maximum energy equal to zero at $R_{\min} = (3^{3/2}/4)(L^2/m)$, where m and L are the mass of the muon and the absolute value of the projection of its angular momentum on the proton–electron axis, respectively. In the limit of R approaching infinity, the energy asymptotically approached $-(1/2)(m/L^2)$. The muon revolution frequency was very close to its maximum value m/L^3 (which is the Kepler frequency for the corresponding muonic hydrogen atom) for almost all values of R.

In the present paper we studied the general case of $Z > 1$ using the same methods as in ref. 1, as follows. The equation for the effective potential energy is the same as (4) from ref. 1

Fig. 1. A muon rotating in a circle perpendicular to and centered at the axis connecting the nucleus and the electron.



$$U_{\mathrm{eff}}(z,\rho) = \frac{L^2}{2m\rho^2} - \frac{Z}{\sqrt{z^2+\rho^2}} - \frac{Z'}{\sqrt{(R-z)^2+\rho^2}} \tag{1}$$

where (ρ, φ, z) are the cylindrical coordinates, m is the mass of the muon (in atomic units $m = 206.768\,274\,6$), Z and Z' are the charges of the effective nuclei (in our case, $Z' = -1$), and R is the distance between the effective nuclei. Using the scaled quantities defined by (5) from ref. 1,

$$w = \frac{z}{R} \qquad v = \frac{\rho}{R} \qquad \varepsilon = -ER \qquad \ell = \frac{L}{\sqrt{mR}} \qquad r = \frac{mR}{L^2} \tag{2}$$

we get the following equation for the scaled energy of the muon:

$$\varepsilon = \frac{Z}{\sqrt{w^2+p}} - \frac{1}{\sqrt{(1-w)^2+p}} - \frac{\ell^2}{2p} \tag{3}$$

where $p \equiv v^2$ is the squared scaled radial coordinate. Then we require the derivative of the scaled energy with respect to the axial and radial coordinates (w, p) to vanish at equilibrium, which gives us two equations accordingly

$$p = w^{2/3}(w-1)^{2/3}\frac{Z^{2/3}(w-1)^{4/3} - w^{4/3}}{(w-1)^{2/3} - Z^{2/3}w^{2/3}} \tag{4}$$

$$\ell^2 = p^2\left\{\frac{Z}{(w^2+p)^{3/2}} - \frac{1}{[(1-w)^2+p]^{3/2}}\right\} \tag{5}$$

Because the left-hand sides of (4) and (5) are always positive, this imposes conditions on the equilibrium range. For $Z = 1$, it was $w < 0$. In the case of $Z > 1$ it is not half-infinite; it has a lower limit

$$-\frac{1}{Z-1} < w < 0 \tag{6}$$

The analysis of (4) and (5) also shows that there are no equilibrium points for $w > 0$. We substitute the value of ℓ^2 from (5) into (3) and then the value of p from (4) into the resulting equation, obtaining $\varepsilon(w)$ — the scaled energy of the muon dependent on the scaled internuclear coordinate, w, for a given value of Z. As in ref. 1, it is convenient to use the substitution

$$\gamma = \left(1 - \frac{1}{w}\right)^{1/3} \tag{7}$$

which allows for a more compact form of the equation for ε (as well as other equations). Also, from scaling (2) we have $r = 1/\ell^2$. Finally, we use the scaled energy $\varepsilon_1 = \varepsilon/r$, which has the same scaling as r. Thus, we obtain the parametric dependence, $\varepsilon_1(r)$, with the parameter w (or γ), which yields the energy terms for a given Z. We present the parametric equations

$$\varepsilon_1 = \frac{(Z^{2/3}\gamma^4 - 1)^2[Z^{2/3}\gamma(\gamma^3 + 2) - 2\gamma^3 - 1]}{2(\gamma^3 - 1)(\gamma^3 + 1)^3} \tag{8}$$

$$r = \frac{\sqrt{(\gamma^6 - 1)^3(\gamma^2 - Z^{2/3})}}{\gamma(Z^{2/3}\gamma^4 - 1)^2} \tag{9}$$

Figure 2 presents the energy terms for the values of $Z = 2$, 3, 4, and 5.

Let us summarize the differences from the case of $Z = 1$. The equilibrium range in the case of $Z > 1$ is not half-infinite (as it was for $Z = 1$), but rather it has a lower limit. In terms of the scaled internuclear coordinate $w = z/R$, it is given by (6), which yields $-\infty < w < 0$ for $Z = 1$. Thus, the equilibrium range is drastically reduced in the case of $Z > 1$: for $Z = 2$, it is $-1 < w < 0$, for $Z = 3$ it is $-1/2 < w < 0$, and so on.

Each curve in Fig. 2 represents two coinciding energy terms: there is a double degeneracy with respect to the sign of the projection of the muon angular momentum on the internuclear axis. This was also the case for $Z = 1$. However, in distinction to the case of $Z = 1$, now the energy terms start not at some nonzero R_{min}, but at $R_{min} = 0$. This is because the muon would be bound even if the nucleus and electron are at the same point, because their total charge is positive. Another distinction from the case of $Z = 1$ is that the energy of the muon at $R = 0$ is not zero, but rather it is $-(Z - 1)^2[m/(2L^2)]$. As R increases, the energy diminishes approaching $-Z^2m/(2L^2)$ at $R \to \infty$. The binding energies for $Z > 1$ are much greater than for $Z = 1$ at any finite R for the same reason as stated earlier: for $Z = 1$ the total charge of the nucleus–electron subsystem is zero, so that the muon is only relatively weakly bound compared to the case of $Z > 1$, where the total charge of the nucleus–electron subsystem is positive.

The following note might be useful. Each curve in Fig. 2 represents two degenerate classical energy terms of "the same symmetry". (In physics of diatomic molecules, the terminology "energy terms of the same symmetry" means the energy terms of the same projection of the angular momentum on the internuclear axis.) For a given R and L, the classical energy, E, takes only one *discrete* value. However, as L varies over a *continuous* set of values, so does the classical energy, E (as it should in classical physics).

The behavior of the muon revolution frequency is similar to the case of $Z = 1$, but now its maximum value is Z^2m/L^3, which is the Kepler frequency for the muonic hydrogen-like ion of nuclear charge Z. In Appendix A we present the analysis of the stability of the muonic motion using the same technique as in ref. 1 by generalizing it for $Z > 1$. We found that the muonic motion is stable, just as it was for $Z = 1$.

3. Electronic motion and the validity of the scenario

The condition for the separation of the rapid and slow subsystems was that the ratio Ω/ω of the muon and electron revolution frequencies should be much greater than unity. For this to be valid, the case of $Z = 1$ required the ratio L/M of the muon and electron angular momenta to be noticeably greater than 20. Calculations show that as Z increases, the required ratio L/M increases to maintain the same condition for Ω/ω.

Using the same method as in Sect. 3 of ref. 1, we obtained the following equations for the frequency of the muon (Ω) and of the electron (ω)

$$\Omega = \frac{m}{L^3}v \qquad v = \frac{(Z^{2/3}\gamma^4 - 1)^3}{(\gamma^3 - 1)(\gamma^3 + 1)^3} \tag{10}$$

$$\omega = \frac{m^{3/2}}{L^3}\frac{\ell_e}{r^{3/2}} \tag{11}$$

Fig. 2. Classical energy terms: the scaled energy, $\varepsilon_1 = (L^2/m)E$, versus the scaled internuclear distance, $r = (m/L^2)R$, for Z = (solid curve) 2, (dashed curve) 3, (dot-dashed curve) 4, and (dotted curve) 5.

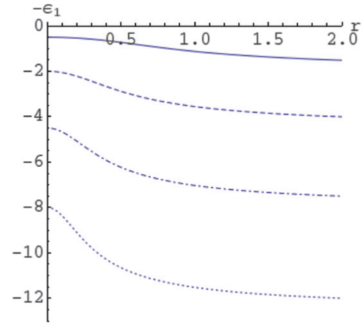
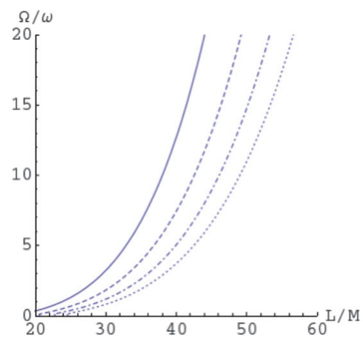


Fig. 3. The ratio of the muon and electron revolution frequencies Ω/ω versus the ratio of the muon and electron angular momenta L/M for $Z = 2$ (solid curve), $Z = 3$ (dashed curve), $Z = 4$ (dot-dashed curve) and $Z = 5$ (dotted curve).



where r is given by (9) and ℓ_e is the equilibrium value of the scaled electron angular momentum $M/R^{1/2}$ (M is the electron angular momentum in atomic units)

$$\ell_e = \sqrt{Z}\sqrt{1 - \frac{(1 - \gamma)^2\sqrt{1 + \gamma + \gamma^2}}{(1 - \gamma + \gamma^2)^{3/2}}} \tag{12}$$

For $Z = 1$, (10)–(12) transform into those given in ref. 1. Combining these equations, we express the ratio of the muon's and electron's frequencies as well as angular momenta

$$\frac{\Omega}{\omega} = \frac{1}{\sqrt{m}}\frac{vr^{3/2}}{\ell_e} \tag{13}$$

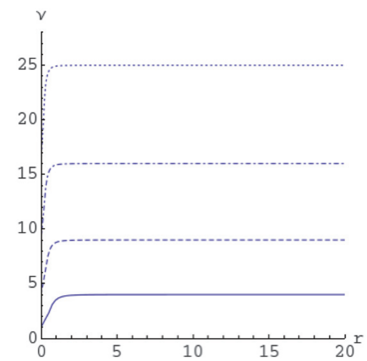
$$\frac{L}{M} = \frac{\sqrt{m}}{\ell_e\sqrt{r}} \tag{14}$$

We plot the two ratios for selected values of Z (shown in Fig. 3). We see that, for example, for $Z = 5$, the ratio L/M must be noticeably greater than 34 to satisfy the validity condition $\Omega/\omega \gg 1$.

4. Red shift of spectral lines compared to the muonic hydrogen-like ions of the nuclear charge *Z*

The muon, rotating in a circular orbit at frequency $\Omega(R)$, should emit a spectral line with this frequency. The maximum value

Fig. 4. The scaled muon revolution frequency, $v = (L^3/m)\Omega$, versus the scaled internuclear distance, $r = (m/L^2)R$, for Z = (solid curve) 2, (dashed curve) 3, (dot-dashed curve) 4, and (dotted curve) 5.



$\Omega_{max} = Z^2m/L^3$ is the Kepler frequency for the muonic hydrogen-like ion of the nuclear charge, Z. In Fig. 4 we present the plots for the scaled muon frequency, v (given in (10)) versus the scaled nucleus–electron distance, r, for Z = 2, 3, 4, and 5.

Therefore, as in the case Z = 1 [1], the spectral lines emitted by the muon experience a red shift compared to the spectral lines of the corresponding muonic hydrogen-like ions. The relative red shift, defined as

$$\delta = \frac{\Omega_{max} - \Omega}{\Omega} \tag{15}$$

will be represented here as

$$\delta = \frac{1}{v(\gamma)} - 1 \tag{16}$$

Using (16) and (14) as the parametric equations with the parameter γ, we plot the relative red shift with respect to the ratio of muon and electron angular momenta for some typical values of Z ($\lg x = \log_{10} x$) in Fig. 5.

Using (16) and (13) as the parametric equations with the parameter γ, we plot the relative red shift with respect to the ratio of muon and electron frequencies for the same values of Z in Fig. 6.

It is seen that the relative red shift decreases as Z increases. However, it remains within the spectral resolution of available spectrometers (10^{-4}–10^{-5}) at least up to Z = 5.

We present specific examples for several values of Z summarized in Table 1. In the last row, "critical value of L/M" is the value that must be noticeably exceeded for the separation of the rapid and slow subsystems to be valid. For example, for Z = 4, the ratio of the muon and electron revolution frequencies is much greater than unity if the ratio L/M of the muon and electron angular momenta is noticeably greater than 30.

5. Conclusion

We studied muonic–electronic quasi molecules μ*Ze* with $Z > 1$. We demonstrated that in this case, just as in the previously studied case of Z = 1, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem; a result that might seem counterintuitive. In other words, the muon rapidly revolves in a circular orbit about the axis connecting the proton and electron while this axis slowly rotates following a relatively slow electronic motion. For this to be valid, the ratio of the muonic and electronic angular momenta should be slightly greater than in the case of Z = 1.

Fig. 5. Dependence of the relative red shift, δ, of the spectral lines of the quasi molecule μ*Ze* on the ratio of the muon and electron angular momenta for Z = (solid curve) 2, (dashed curve) 3, (dot-dashed curve) 4, and (dotted curve) 5.

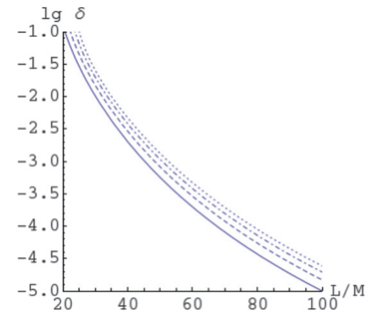
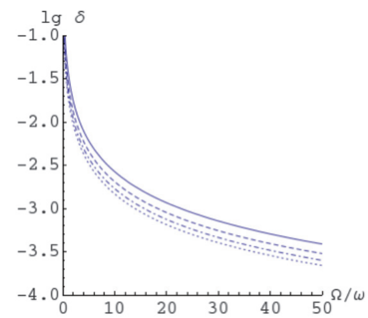


Fig. 6. Dependence of the relative red shift, δ, of the spectral lines of the quasi molecule μ*Ze* on the ratio of the muon and electron frequencies for Z = (solid curve) 2, (dashed curve) 3, (dot-dashed curve) 4, and (dotted curve) 5.



We used a classical analytical description to find the energy terms of such a system (i.e., dependence of the energy of the muon on the distance between the proton and electron). We found that there is a double-degenerate energy term. We demonstrated that it corresponds to a stable motion. The binding energies for $Z > 1$ are much greater than for Z = 1 at any finite value of the nucleus–electron distance. This is because for Z = 1 the total charge of the nucleus–electron subsystem is zero, so that the muon is only relatively weakly bound compared to the case of $Z > 1$, where the total charge of the nucleus–electron subsystem is positive.

Then we unfroze the slow subsystem and analysed a slow revolution of the axis connecting the proton and electron. The slow subsystem can be treated as a modified "rigid rotator" consisting of the electron, the nucleus, and the ring over which the muon charge is uniformly distributed, all distances within the system being fixed. We derived the condition required for the validity of the separation into the rapid and slow subsystems.

Finally we showed that the spectral lines, emitted by the muon in the quasi molecule μ*Ze*, experience a red shift compared to the spectral lines of the corresponding muonic hydrogen-like ion of the nuclear charge, Z. This shift, which is the effect of the formation of the quasi molecule, is counted from the well-known positions of the spectral lines of the muonic hydrogen-like ion, so that it is well-defined and suited for experimental observation. The relative red shift decreases as Z increases, but remains within the spectral resolution of available spectrometers at least up to Z = 5. Observing this red shift should be one of the ways to detect the formation of the quasi molecules μ*Ze*.

It should be emphasized that circular states of atomic and molecular systems is an important subject in its own right. They have been extensively studied both theoretically and experimentally

Table 1. Selected parameters of the system for $Z = 2, 3, 4, 5$, and for arbitrary Z.

Z	2	3	4	5	Arbitrary
Muon equilibrium range	$-1 < w < 0$	$-1/2 < w < 0$	$-1/3 < w < 0$	$-1/4 < w < 0$	$-1/(Z-1) < w < 0$
Muon maximum energy (at $R = 0$)	$-(1/2)(m/L^2)$	$-2(m/L^2)$	$-(9/2)(m/L^2)$	$-8(m/L^2)$	$-[(Z-1)^2/2](m/L^2)$
Muon minimum energy (at $R \to \infty$)	$-2(m/L^2)$	$-(9/2)(m/L^2)$	$-8(m/L^2)$	$-(25/2)(m/L^2)$	$-(Z^2/2)(m/L^2)$
Maximum value of muon frequency	$4(m/L^3)$	$9(m/L^3)$	$16(m/L^3)$	$25(m/L^3)$	$Z^2(m/L^3)$
Critical value of L/M	25	28	30	30	Nonanalytical

(see, e.g., refs. 7–11 and references therein). In the present paper, just as in our previous paper [1], we used circular states just to get the message across and to stimulate further studies of the quasi molecules μZe.

We also note that it is possible to go beyond the circular states. From the general analytical results of ref. 12, it follows that there are *stable helical states*. For the quasi molecule μZe this means that the muon trajectory would be a helix on the surface of a cone, with the axis coinciding with the line connecting the proton and the electron. In this helical state, the muon, while rapidly spiraling on the surface of the cone, oscillates between two end-circles that result from cutting the cone by two parallel planes (very close to each other) perpendicular to its axis. Thus, from the circular state, under a fluctuation the quasi molecule would switch to such a stable helical state.

Last, but not least: the theoretical approach based on the separation of rapid and slow subsystems required the muon to be in a state of a high angular momentum. Luckily, the experimental methods to create, for example, muonic hydrogen atoms, μp, lead to the muon being in a highly excited state (see, e.g., refs. 13 and 14). We also mention ref. 15 where it has been shown, in particular, that the distribution of the muon principal quantum number in muonic hydrogen atoms peaks at larger and larger values with the increase of the energy of the muon incident on electronic hydrogen atoms. This justified the classical treatment in our paper [1]. The situation for the systems with Z > 1 should be similar, thus justifying the classical treatment in the present paper.

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Appendix A. The analysis of the stability of the muonic motion

To analyse the stability of the muonic motion, we write down the Hamiltonian of the muonic motion in the cylindrical coordinates:

$$H = \frac{1}{2}\left(p_\rho^2 + p_z^2 + \frac{p_\varphi^2}{\rho^2}\right) + U(z,\rho) \tag{A1}$$

where the potential energy of the muon in the presence of the slow subsystem "nucleus–electron" is

$$U(z,\rho) = -\frac{Z}{\sqrt{z^2+\rho^2}} - \frac{Z'}{\sqrt{(R-z)^2+\rho^2}} \tag{A2}$$

where $Z' = -1$. Because φ is a cyclic coordinate, the corresponding momentum is conserved

$$|p_\varphi| = \text{const.} = L \tag{A3}$$

With this substituted into (A1), we obtain the Hamiltonian for the z- and ρ-motions

$$H_{z\rho} = \frac{1}{2}(p_\rho^2 + p_z^2) + U_{\text{eff}}(z,\rho) \tag{A4}$$

where U_{eff} is given by (1). With the scaling formulas as in (2), U_{eff} can be rewritten as

$$U_{\text{eff}} = -\frac{Z}{R}\varepsilon \tag{A5}$$

with ε given by (3). Then we seek equilibrium points in the $z\rho$- (or wv-) space by equating the derivatives of the effective potential energy with respect to both coordinates to zero

$$\frac{\partial\varepsilon}{\partial w} = 0 \qquad \frac{\partial\varepsilon}{\partial v} = 0 \tag{A6}$$

The first relation in (A6) gives us (4), which determines a line in the (w, v) plane where the equilibrium points are located, while the second relation in (A6) gives us (5), which determines the equilibrium value of the muon angular momentum. Thus, for a given value of the axial coordinate, w, in the allowed range (given by (6)), there exist the equilibrium values of the orbit radius, v_0 (square root of the right-hand side of (4)) and the angular momentum, ℓ_0 (square root of the right-hand side of (5) with p substituted from (4)).

Next, we consider small deviations from equilibrium

$$\delta w = w - w_0 \qquad \delta v = v - v_0 \tag{A7}$$

(where w_0 is the point where the equilibrium values are taken; later we drop the index from w) and expand the effective potential energy, ε, in terms of these small deviations

$$\varepsilon \approx \varepsilon_0 + \frac{1}{2}\frac{\partial^2\varepsilon}{\partial w^2}\delta w^2 + \frac{1}{2}\frac{\partial^2\varepsilon}{\partial v^2}\delta v^2 + \frac{\partial^2\varepsilon}{\partial w\partial v}\delta w\delta v \qquad \text{(A8)}$$

where ε_0 is the potential energy at equilibrium. The second derivatives evaluated at equilibrium (at any allowed w, $v = v_0$ and $\ell = \ell_0$ as given by (4) and (5)) are as follows:

$$\frac{\partial^2\varepsilon}{\partial w^2} = \frac{1}{\left(w^2 + v_0^2\right)^{3/2}}\left(\frac{1}{1-w} - \frac{3wP}{Q^2}\right) \qquad \text{(A9)}$$

$$\frac{\partial^2\varepsilon}{\partial v^2} = \frac{1}{\left(w^2 + v_0^2\right)^{3/2}}\left(\frac{1}{1-w} + \frac{3wP}{Q^2}\right) \qquad \text{(A10)}$$

$$\frac{\partial^2\varepsilon}{\partial w\partial v} = \frac{1}{\left(w^2 + v_0^2\right)^{3/2}}\frac{3wv_0(2w-1)}{Q^2} \qquad \text{(A11)}$$

where the quantities P and Q are defined as follows:

$$P = w(1-w) + v_0^2 \qquad Q = \sqrt{\left(w^2 + v_0^2\right)\left[(1-w)^2 + v_0^2\right]} \qquad \text{(A12)}$$

Because generally the derivative in (A11) is not zero, a rotation of the reference frame is required to transform the potential energy to normal coordinates, diagonalizing the matrix of the second derivatives (A9)–(A11)

$$\delta w' = \delta w\cos\alpha + \delta v\sin\alpha \qquad \delta v' = -\delta w\sin\alpha + \delta v\cos\alpha \qquad \text{(A13)}$$

where α is the angle of rotation. It can be found that

$$\mathrm{tg}2\alpha = 2\frac{\partial^2\varepsilon}{\partial w\partial v}\left(\frac{\partial^2\varepsilon}{\partial w^2} - \frac{\partial^2\varepsilon}{\partial v^2}\right)^{-1} = \frac{(1-2w)v_0}{P} \qquad \text{(A14)}$$

In the normal coordinates, the expansion of ε takes the form

$$\varepsilon \approx \varepsilon_0 + \frac{1}{2}\delta w'^2\omega_-^2 + \frac{1}{2}\delta v'^2\omega_+^2 \qquad \text{(A15)}$$

where

$$\omega_\pm = \sqrt{\frac{1}{\left(w^2 + v_0^2\right)^{3/2}}\left(\frac{1}{1-w} \pm \frac{3w}{Q}\right)} \qquad \text{(A16)}$$

ω_+ is real when

$$\frac{1}{1-w} + \frac{3w}{Q} \geq 0 \qquad \text{(A17)}$$

Under this condition, ω_+ is the scaled frequency of small oscillations about the equilibrium in the direction of the normal coordinate $\delta v'$ and ω_- is real when

$$Q \geq 3w(1-w) \qquad \text{(A18)}$$

Under this condition, ω_- is the scaled frequency of small oscillations about the equilibrium in the direction of the normal coordinate $\delta w'$.

Thus, if both conditions (A17) and (A18) are satisfied, the potential energy has a two-dimensional minimum at the equilibrium point, so that the equilibrium is stable.

Condition (A18) is always satisfied for any allowed w. From the allowed range of w, (6), it is seen that $w < 0$, so the left-hand side, Q, which is always nonnegative, will be always greater than the negative right-hand side. Condition (A17) may be rewritten as

$$\left(w^2 + v_0^2\right)\left[(1-w)^2 + v_0^2\right] - 9w^2(1-w)^2 \geq 0 \qquad \text{(A19)}$$

Using the value of $v_0^2 = p$ from (4) and applying the γ-substitution given in (7), condition (A19) becomes

$$Z^{2/3}(1 + 16\gamma^6 + \gamma^{12}) - 9\gamma^4(Z^{4/3} + \gamma^4) \geq 0 \qquad \text{(A20)}$$

In the allowed range $Z^{1/3} < \gamma < +\infty$ (which is (6) after γ-substitution (7)), this condition is always satisfied. Therefore, the muonic motion is stable.