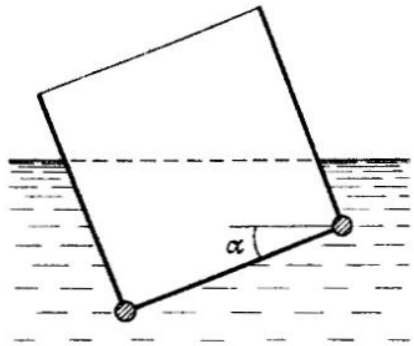
Problem. The cubic box is floating submerged by half in the water and tilted by angle α. Determine the masses of the two lower edges (marked in the figure, the ones that are perpendicular to the plane of the figure). Ignore the mass of other parts of the box. The density of water is ρ, the length of each edge of the box is *a*.



Solution. Let the mass of the left edge be m_1 and of the right edge m_2. Since the box is half-submerged, the Archimedes force, balancing the total weight, is $\rho g a^3/2$:

$$m_1 + m_2 = \frac{\rho a^3}{2} \tag{1}$$

(we divided the equation by g). The second equation should come from the balance of the torques. The torques acting on the front face and the back face cancel due to symmetry, and we only consider the torques acting on the left, bottom and right faces, which we label M_1, M_2 and M_3 accordingly. The torques will be calculated with edge m_1 being the pivot. Let the depth of the m_1 edge be H and of the m_2 edge be h. The box is submerged by half, so its center should be at the surface level. From the geometry we have

$$H = \frac{a}{2}(\cos\alpha + \sin\alpha), \quad h = \frac{a}{2}(\cos\alpha - \sin\alpha) \tag{2}$$

Let the y-axis be vertically down with the 0 on the water surface. At the depth y, the pressure is $\rho g y$. Choose a horizontal strip of the left face at this depth: its length is a and its height is dy, so its width is $dy/\cos\alpha$. The force acting on it is equal to the pressure $\rho g y$ times the area of the strip, and the moment arm is parallel to the face surface and has length $(H - y)/\cos\alpha$. Therefore, the elementary torque acting on the horizontal strip of the left face of vertical height dy at the depth y is

$$dM_1 = -\rho g y \left(a \frac{dy}{\cos\alpha}\right) \frac{H - y}{\cos\alpha} \tag{3}$$

the minus sign showing that the torque is clockwise with respect to m_1. The total torque on the left face will then be

$$M_1 = \int_0^H dM_1 = -\frac{1}{6}\rho g a \frac{H^3}{\cos^2\alpha} = -\frac{\rho g a^4}{48\cos^2\alpha}(\cos\alpha + \sin\alpha)^3 \tag{4}$$

where we substituted H from (2). We use the same procedure to find the torques on the bottom and on the right face (both of them are counterclockwise):

$$dM_2 = \rho g\, y \left(a \frac{dy}{\sin\alpha}\right) \frac{H-y}{\sin\alpha} \tag{5}$$

$$M_2 = \int_h^H dM_2 = \frac{1}{6}\rho g a \frac{(H-h)^2}{\sin^2\alpha}(H+2h) = \frac{\rho g a^4}{12}(3\cos\alpha - \sin\alpha) \tag{6}$$

$$dM_3 = \rho g\, y \left(a \frac{dy}{\cos\alpha}\right) \frac{h-y}{\cos\alpha} \tag{7}$$

$$M_3 = \int_0^h dM_3 = \frac{1}{6}\rho g a \frac{h^3}{\cos^2\alpha} = \frac{\rho g a^4}{48\cos^2\alpha}(\cos\alpha - \sin\alpha)^3 \tag{8}$$

The fourth non-zero torque is that of the weight $m_2 g$:

$$M_{m_2 g} = -m_2 g a \cos\alpha \tag{9}$$

All the torques should sum up to zero:

$$\begin{aligned} &-\frac{\rho g a^4}{48\cos^2\alpha}(\cos\alpha + \sin\alpha)^3 + \frac{\rho g a^4}{12}(3\cos\alpha - \sin\alpha) + \\ &+ \frac{\rho g a^4}{48\cos^2\alpha}(\cos\alpha - \sin\alpha)^3 - m_2 g a \cos\alpha = 0 \end{aligned} \tag{10}$$

from where we find the solution for the second edge:

$$m_2 = \frac{\rho a^3}{24}(6 - 5\,\mathrm{tg}\,\alpha - \mathrm{tg}^3\alpha) \tag{11}$$

and, using (1), find the solution for the first edge:

$$m_1 = \frac{\rho a^3}{24}(6 + 5\,\mathrm{tg}\,\alpha + \mathrm{tg}^3\alpha) \tag{12}$$

References

[1] O. Ya. Savchenko "Zadachi po fizike" ("Problems in physics"), 2nd ed., Moscow: Nauka, 1988 (in Russian); problem 4.2.19.