Archimedes' Law

Pressure. If a force is acting on a certain area and is uniformly distributed on it, the pressure is a physical quantity which equals the magnitude of the force per unit area:

$$p = \frac{F}{S} \quad .$$

The unit of pressure in SI is *Pascal* (Pa) which equals one Newton per square meter (N/m^2). In everyday life, 1 Pa is a very small amount of pressure. The atmospheric pressure, which varies with height, is about 10^5 Pa or 100 kPa; the accepted standard is 101 325 Pa, which is also a unit of pressure called the standard atmosphere (atm).

Let's calculate the pressure in a liquid depending on the depth. Let's place a horizontal area S at the depth h in a liquid of density ρ. The "pole" of liquid above the area, which exerts the force on it, has the volume Sh, thus its mass is ρSh and its weight is ρgSh. The pressure on our area is the weight of the liquid pole above it divided by the area:

$$p = \frac{\rho g S h}{S} = \rho g h \quad .$$

The pressure under the liquid depends only on the density and depth and does not depend on the area. This may seem counterintuitive, especially if you have a big barrel and a cup filled with water of the same height; despite there is much more water in the barrel than in the cup, the pressure on their bottoms will be the same.

Archimedes' force. Consider an object submerged in a liquid of density ρ (Fig. 1).

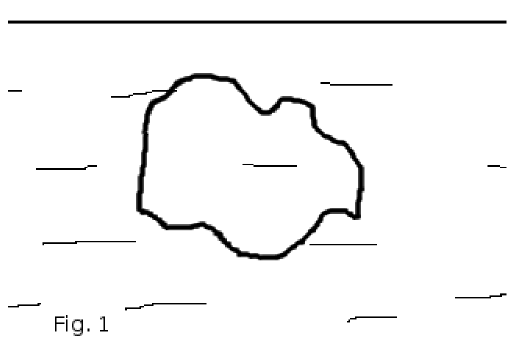

Fig. 1

Since the pressure under the liquid increases with depth, it can be intuitively understood that the bottom part of the object will experience the upward force from the liquid greater in magnitude than the downward force from the liquid on the top of the object. This will result in the net force upwards. It is called Archimedes' force.

We can find it in two ways: by a thought experiment and by a rigorous mathematical proof.

Thought experiment. Archimedes' force is produced by the pressure under the liquid. Let us now look at the object. It takes some volume *V* under the liquid. Now let us replace this object with the same liquid in this volume. It is obvious that this volume of liquid (which was previously occupied by our object) will not move, it will be in equilibrium (otherwise the liquid would be in constant motion). Therefore, Archimedes' force on this volume will equal the weight of it, which is its mass ρV times g, i.e. $\rho g V$. Now replace the liquid in this volume back by our object. The pressure will be the same on the same points of the volume; therefore, Archimedes' force, which is produced by the pressure, should not change. It will stay the same $\rho g V$ for any object.

Conclusion: $F_A = \rho g V$, where ρ is the density of the liquid and V is the volume of the object submerged.

Mathematical proof. We can calculate F_A straightforwardly "from scratch". For this we will need vector calculus.

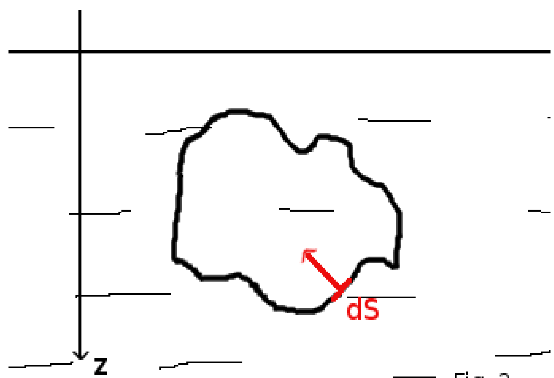

Fig. 2

Consider the same object submerged into a liquid. Let us choose a z-axis vertically downward, as in Fig. 2, so the pressure will depend on it as $p(z) = \rho g z$. Let us choose an infinitesimal area *dS* on the object. Its *vector area* (http://en.wikipedia.org/wiki/Vector_area) is directed outwards, and the force from the liquid is directed into the object, so the infinitesimal force on this area will have a minus sign:

$$d\vec{F} = -p\,d\vec{S} = -\rho g z\,d\vec{S} \quad .$$

Now we just need to sum up all of these small forces, i.e. integrate over all surface *S* of the object.

$$\vec{F}_A = \oint_S d\vec{F} = -\rho g \oint_S z\,d\vec{S} \quad .$$

Using the identity following from Gauss' law (http://en.wikipedia.org/wiki/Divergence_theorem),

$$\oint_S f\,d\vec{S} = \int_V \nabla f\,dV \quad ,$$

we have

$$\vec{F}_A = -\rho g \int_V \nabla z\,dV = -\rho g \int_V \begin{pmatrix}0\\0\\1\end{pmatrix} dV = -\rho g \begin{pmatrix}0\\0\\1\end{pmatrix} \int_V dV = -\rho g V \begin{pmatrix}0\\0\\1\end{pmatrix} = -\rho g V \vec{e}_z \quad ,$$

so the magnitude of the force is $\rho g V$ and the direction is negative with respect to our downward z-axis, which means F_A is upwards.

Sometimes not the whole volume of the object is submerged. Obviously, the pressure from the liquid will be acting only on the volume under the liquid (V_u), so we can rewrite the formula for Archimedes' force as $F_A = \rho_0 g V_u$, where ρ_0 is the density of the liquid and V_u is the volume submerged.

Problem 1. Two balls connected by a string are floating in the water as shown on Fig 3. The volume of each ball is V and the lower one is three times as heavy as the upper one. The upper ball is submerged by half its volume. Find the tension of the string.

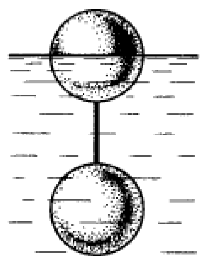

Fig. 3

Solution. From the equilibrium of the forces on each ball we have:
lower ball: $3mg = T + F_{A1}$; upper ball: $mg + T = F_{A2}$. Archimedes force on the lower ball is $F_{A1} = \rho_0 g V$, on the upper one it is $F_{A2} = \frac{1}{2}\rho_0 g V$ (ρ_0 is the density of water). Substituting them into the two Newton's equations above, we have:

$$\begin{cases} 3mg = T + \rho_0 g V \\ mg = \frac{1}{2}\rho_0 g V - T \end{cases} \text{, then } 3\left(\frac{1}{2}\rho_0 g V - T\right) = T + \rho_0 g V \text{, and } T = \frac{1}{8}\rho_0 g V \ .$$

Problem 2. A ball of density ρ is at the depth H under water, whose density is ρ_0, and $\rho < \rho_0$. When the ball is released, what maximum height above the water surface will it achieve? Neglect viscosity of the water and air resistance.

Solution. Under water, the forces acting on the ball are gravity and Archimedes' force:

$$ma = F_A - mg = \rho_0 g V - mg \ .$$

The ball's density is $\rho = \frac{m}{V}$, so $V = \frac{m}{\rho}$, and $ma = \rho_0 g \frac{m}{\rho} - mg$, or $a = \left(\frac{\rho_0}{\rho} - 1\right) g$ – acceleration of the ball under water.

At the moment it goes out of the water, it will have speed v such that $v^2 = 2aH = 2\left(\frac{\rho_0}{\rho} - 1\right) g H$. With this speed, it will achieve the maximum height h such that $v^2 = 2gh$. Setting these two equal, we have $h = H\left(\frac{\rho_0}{\rho} - 1\right)$.