Rotational Dynamics

Since we started looking at what happens to kinematics on a circular track and found complete correspondence to linear kinematics, we shall naturally proceed to looking at dynamics and seeing if the correspondence with dynamics of linear motion would still be this great.

The first difficulty arises from the fact that the concept of the material point (an object whose form and shape are insignificant for the solution), the model we have been using from the beginning of the course until now, is not applicable anymore. We are interested in the properties of rotating objects, and one cannot rotate a point about itself. Now all objects "suddenly" acquire (in our derivations) their form and shape, and we have to deal with them.

Due to this change, real objects have more degrees of freedom than material points. Consider a straight uniform stick with its center at the origin. Besides moving in three directions, it can also rotate with its center still being at the origin.

Forces, whose point of application did not matter for solving Newton's law problems, now produce different effects when applied to different points of the object, even if their magnitudes and directions stay the same. For example, look at the figure below.

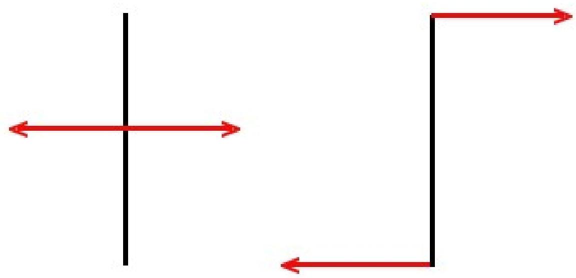

Fig. 1

The black hard rod, previously still, is acted upon by two equal and opposite forces (shown in red). Newton's law for the material point would say that in this case the net force is zero and the point would stay at rest. But does the rod stay at rest? From common experience, it's not hard to see that in the configuration on the left it would rest, but on the right it would not! Indeed, it would start rotating about its center. What is also worth noting is that its center would still be at the same point. From this observation we conclude that 1) zero net force is not sufficient for a real object to stay at rest, and 2) zero net force would still make the center of the object stay at rest, but there is *something* that depends on the point of application that can make the object rotate.

We still need to clear up some points. First of all, we assume that the rod is hard, i.e., you cannot bend or deform it under the conditions we consider. Such a model is called a *rigid body*. Rigid bodies, as well as material points, do not exist in the world (anything can be bent or deformed), but most hard objects under appropriate interactions can be considered rigid bodies. How can we mathematically define a rigid body? If we take any two points of a rigid body, then the distance between them must stay constant. If we draw the radii-vectors of these two random points *i* and *j* (from the origin to each one) and call them $\vec{r}_i$ and $\vec{r}_j$, then the distance $|\vec{r}_i - \vec{r}_j| = const$ for any *i* and *j* on the rigid body.

Second, we have been talking about the *center* of the rod in the figure above. While the center of the uniform rod is located unambiguously at its half-length, where is the "center" of, say, a pear-shaped object or a randomly-shaped stone? Which point of these objects will stay still when they are acted upon by forces which sum up to zero? This point is called the *center of mass* (or, sometimes, the center of inertia), and its position for any random object or object system is defined mathematically:

$$\vec{r}_c = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \quad ,$$

where m_i and $\mathbf{r}_i$ are the mass and radius-vector of the i^{th} component of the system (for example, three little balls at each apex of a triangle). It can be noticed that the denominator is the total mass of the system. For continuous objects (for example, a rod, which consists of infinity of infinitesimally small "components"), summations become integrals:

$$\vec{r}_c = \frac{\int_V dm\,\vec{r}}{\int_V dm} = \frac{1}{m}\int_V dm\,\vec{r} = \frac{1}{m}\int_V \rho\,\vec{r}\,dV \quad ,$$

where ρ is the density and integration is performed on the whole volume. For most symmetrical uniform objects (rod, flat disk, sphere, equilateral triangle), the center of mass coincides with the geometrical center.

Thus, Newton's equations, which were true for the material point, are also true for the *center of mass* of any rigid body. If the net force on the rigid body is zero, the acceleration of its center of mass is zero. Now, what is that *something*, mentioned above, which makes rigid bodies rotate?

This physical quantity clearly depends on the point of application of the forces (as seen from Fig. 1). Also, from common sense, it must depend on the magnitude of the forces (the harder you pull, the faster the rotation will increase). Would it be the product of the force and some distance from the point of application? Yes! This quantity is called *torque of the force* and equals the magnitude of the force times the *moment arm* (the distance from the pivot point to the line on which the force lies; the pivot may not be attached, it's just enough that the object revolves around that point).

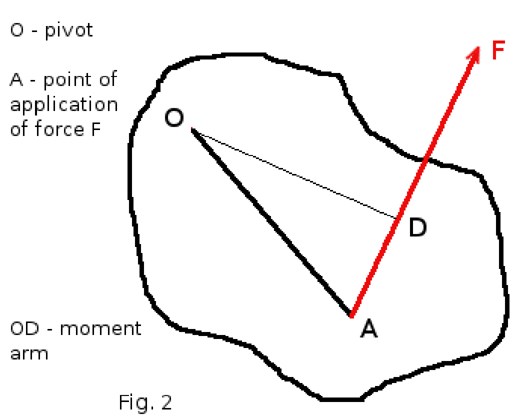

Fig. 2

$M_F = Fd$.

On the left of Fig. 1, both forces are applied in the pivot itself and thus their moment arms are zero, therefore, the torque of both equals zero and we have no rotation. On the right of Fig. 1, each force has a torque equal to the magnitude of either times half-length of the rod, and both torques are clockwise.

In more advanced courses, it is introduced that the torque is a vector quantity $\vec{M}_F = \vec{r} \times \vec{F}$, where **r** is the radius-vector of the point of application of the force **F**. Thus, one can consider projections of torques as positive and negative, as opposed to clockwise and counterclockwise. The torques on Fig. 1 would then be directed into the page (somewhat counterintuitive). This is indispensable when forces are directed in all three dimensions under different angles, but in a two-dimensional setup as we usually have the torque will always be into the paper or out of the paper, so it would suffice if we just use the "clockwise-counterclockwise" notation. The question of which direction to consider positive is a matter of agreement. I would use clockwise as positive here.

Thus, force causes translation, torque causes rotation. More exactly, both cause *change* in both. As the change in translational speed is called acceleration and the change in angular speed is called angular acceleration, we can now conjecture the rotational analogue of second Newton's law. Since $\vec{F} = m\vec{a}$, $M_F = ?\epsilon$, where the question mark represents the analogue of mass. Since bigger mass is what makes an object harder to accelerate, this analogue of mass is what makes an object harder to rotate faster, or angularly accelerate. This analogue is called the *moment of inertia*, and is mathematically defined as $I = \sum_i m_i r_i^2 = \int_V dm\, r^2 = \int_V \rho r^2 dV$, where the sum is for discrete, and the integral is for continuous distributions. Note that the *r* in the definition is the distance to the axis of rotation, not to the point! (You cannot rotate something about a point, only about an axis.) It is obvious that the moment of inertia, like mass, depends only on the internal properties of the object, stays the same for the same object, and does not depend on the object's rotation or any forces applied (just like the mass of the object does not depend on its acceleration). The moment of inertia, like mass, is additive, that is, the moment of inertia of the system equals the sum of moments of inertia of its components.

The moment of inertia of a material point rotated around some axis is obviously mr^2, since all the material is concentrated at the same distance from the axis at the same tiny place. The same is true for a thin ring about its center: its parts are all at the same distance R from its center and thus R^2 will factor out, and the masses will integrate (or sum up to) the total mass m.

The moment of inertia is always *about* some axis. It may be different about a different axis. The moments of inertia of the basic geometrical objects (rods, disks, spheres) are usually given in reference materials about an axis going through the center of mass and perpendicular to the plane if the object is two-dimensional (for three-dimensional objects, like a cylinder, it is usually their axis of symmetry, and for the sphere it does not even matter since it is the most symmetrical 3-D object).

Problem 1. Find the moment of inertia of a rod of mass m and length ℓ about its center and about one of its ends.

Solution. Let us position the rod parallel to the x-axis and choose a small piece dx of the rod. This piece is a material point, so its small moment of inertia equals its mass times distance to the center of rotation squared:

$$dI = dm\, x^2 \quad .$$

From the uniformity of the rod,

$$\frac{dm}{dx} = \frac{m}{\ell} \text{ , thus } dm = \frac{m}{\ell} dx \text{ , and } dI = \frac{m}{\ell} x^2 dx \quad .$$

Now we need to integrate the rod between its ends. For the zero at its center, they will be at $-\ell/2$ and $\ell/2$. Integration gives:

$$I = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dI = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{m}{\ell} x^2 dx = \frac{m}{\ell} \left(\frac{1}{3} \left(\frac{\ell}{2} \right)^3 - \frac{1}{3} \left(-\frac{\ell}{2} \right)^3 \right) = \frac{m\ell^2}{12} \quad .$$

For the left end at zero, that would be

$$I = \int_0^{\ell} dI = \int_0^{\ell} \frac{m}{\ell} x^2 dx = \frac{m}{\ell} \left(\frac{1}{3} \ell^3 - 0 \right) = \frac{m\ell^2}{3} \quad .$$

There is a very useful **Huygens – Steiner theorem**, or parallel axis theorem, which relates the moment of inertia about the axis going through the center of mass and the one about another axis parallel to the first. If the distance between the axes is d and the moment of inertia of some rigid body about the center of mass is I_0, then the moment of inertia about the parallel axis is $I = I_0 + md^2$, where m is the mass of the rigid body. As an exercise, you can prove this theorem.

Let's check if it works for the rod above. For the center of mass $I_0 = \frac{m\ell^2}{12}$ and the distance between the end and the center is $d = \frac{\ell}{2}$, the moment of inertia about the end should be

$I = I_0 + md^2 = \frac{m\ell^2}{12} + m \left(\frac{\ell}{2} \right)^2 = \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$, as we found before by straightforward integration.

Now that we know enough about moment of inertia, we can state the **torque equation –** the law which corresponds to the second Newton's law in rotational dynamics:

$$M_F = I\epsilon \quad ,$$

or, in words, the net torque equals the moment of inertia times angular acceleration.

Problem 2. A string is wound around a pulley, which is a disk of mass m_1. To the other end of the pulley a mass m_2 was attached and released. Find the acceleration of the second mass.

Solution. The Newton's equation for the second mass is $m_2 a = m_2 g - T$, where T is the tension in the string; from here $T = m_2 g - m_2 a$. The only torque-producing force on the disk is T, since whatever force is at its axle has zero torque due to zero moment arm. The torque of the force T on the pulley is

TR, where *R* is the radius of the pulley. So the torque equation for the pulley is $I\epsilon = TR$, or $\frac{m_1 R^2}{2}\epsilon = TR$, where ε is its angular acceleration. Assuming that the string does not slip on the pulley (the parts of the string that touch the pulley's rim move together with it), acceleration *a* of the second mass, which is also the acceleration of the string, equals the acceleration of the pulley's rim, which is εR. From here $\epsilon = \frac{a}{R}$. Substituting the expressions for *T* and ε into the torque equation, we have:

$$\frac{m_1 R^2}{2}\frac{a}{R} = (m_2 g - m_2 a)R \quad , \quad \frac{m_1 a}{2} = m_2 g - m_2 a \quad \text{, and} \quad a = \frac{m_2 g}{\frac{m_1}{2} + m_2} = \frac{2 m_2 g}{m_1 + 2 m_2} \quad .$$

Equilibrium of a rigid body. From the torque equation it can be seen that when the net torque is zero, the angular acceleration must equal zero. Thus, for the equilibrium of a rigid body, not only should all the forces sum up to zero, but also all the torques (see Fig. 1). Since the condition for the forces to sum up to zero usually produces two useful equations (x- and y-projections of $\vec{F} = 0$), for the standard problem on the equilibrium of a rigid body we need three equations (two Newton's and one torque) to solve for the unknowns. In the torque equation, you can choose the pivot point anywhere you want. It is particularly helpful to choose it at the point with most forces, since by this choice their torques will equal zero.

Problem 3. A ladder is leaning against the wall. When the angle between the floor and the ladder is α or less, it starts sliding down. The coefficient of friction between the floor and the ladder is equal to that between the wall and the ladder. Find it.

Solution. Let's consider the ladder on the verge of slipping.

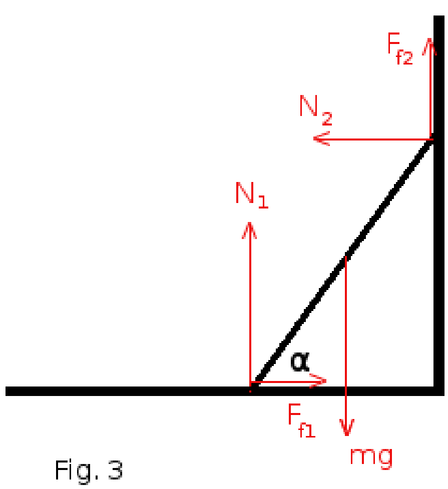

Fig. 3

Let's identify all the forces (there are 5: two friction forces, two normal forces, and gravity) and do the Newton's law on them.

Horizontal: $F_{f1} = N_2$;
Vertical: $N_1 + F_{f2} = mg$.

Since the ladder is on the verge of slipping, $F_{f1}=\mu N_1$ and $F_{f2}=\mu N_2$, where μ is the coefficient of friction that we are trying to find. Plugging it in, we have:

$\mu N_1=N_2, \mu N_2+N_1=mg$. Substituting the first into the second and solving for the normal forces, we have:

$$N_1=\frac{mg}{\mu^2+1}, N_2=\frac{\mu mg}{\mu^2+1} \quad .$$

Now we need to state the third equation – the torque equilibrium (otherwise the data is insufficient to find μ). Let's choose the pivot point at the place the ladder touches the floor. Thus, the torques of N_1 and F_{f1} with respect to this pivot are zero. *mg* is applied at the center of mass, its torque is clockwise (*mg* "wants" to rotate the ladder obviously clockwise around our pivot), and from the triangle we see that its moment arm is $\frac{\ell}{2}\cos\alpha$ (where ℓ is the length of the ladder), so its torque is $mg\frac{\ell}{2}\cos\alpha$. The non-zero torques of the two remaining forces N_2 and F_{f2} are both counterclockwise (if you pull the imaginary strings attached to the ladder in their direction, the ladder will go counterclockwise around its bottom, which we chose as our pivot). Again, from the triangles you calculate their moment arms (for N_2 it is $\ell\sin\alpha$ and for F_{f2} it is $\ell\cos\alpha$), so their torques are $-N_2\ell\sin\alpha$ and $-\mu N_2\ell\cos\alpha$ (as we agreed to consider counterclockwise to be negative). Setting up the torque equation

$$mg\frac{\ell}{2}\cos\alpha-\mu N_2\ell\cos\alpha-N_2\ell\sin\alpha=0 \quad ,$$

or, canceling ℓ and dividing by $\cos\alpha$,

we have $\frac{mg}{2}-\mu N_2-N_2\tan\alpha=0$, or $\frac{mg}{2}=N_2(\mu+\tan\alpha)$.

Substituting $N_2=\frac{\mu mg}{\mu^2+1}$ found earlier, we have:

$\frac{mg}{2}=\frac{\mu mg}{\mu^2+1}(\mu+\tan\alpha)$, *mg* cancels, and we have $\frac{1}{2}=\frac{\mu}{\mu^2+1}(\mu+\tan\alpha)$, or

$$\mu^2+1=2\mu(\mu+\tan\alpha)=2\mu^2+2\mu\tan\alpha \quad .$$

This is quadratic with respect to μ. Gathering the terms in the standard form,

$\mu^2+(2\tan\alpha)\mu-1=0$, we have two solutions:

$$\mu=-\tan\alpha\pm\sqrt{\tan^2\alpha+1}=-\tan\alpha\pm\frac{1}{\cos\alpha} \quad .$$

The minus-sign solution is obviously negative and therefore must be left out. The plus-sign solution,

$$\mu=\frac{1}{\cos\alpha}-\tan\alpha=\frac{1}{\cos\alpha}-\frac{\sin\alpha}{\cos\alpha}=\frac{1-\sin\alpha}{\cos\alpha}$$, is obviously positive and therefore valid.

We can check the solution by considering the angle $\alpha=\frac{\pi}{2}$: if this *is* already the critical angle, i.e., if the ladder starts slipping always unless it is vertical, then μ must be zero. Let's try substituting $\alpha=\frac{\pi}{2}$ in our formula and making sure it outputs zero. Well,

$$\frac{1-\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}=\frac{1-1}{0}=\frac{0}{0}$$, which can be anything. We can either consider the limit by using l'Hôpital's rule and seeing that it is indeed zero, or do a simple trigonometric transformation:

$$\frac{1-\sin\alpha}{\cos\alpha}=\frac{(1-\sin\alpha)(1+\sin\alpha)}{\cos\alpha(1+\sin\alpha)}=\frac{1-\sin^2\alpha}{\cos\alpha(1+\sin\alpha)}=\frac{\cos^2\alpha}{\cos\alpha(1+\sin\alpha)}=\frac{\cos\alpha}{1+\sin\alpha} ,$$

and at $\alpha=\frac{\pi}{2}$, $\frac{\cos\frac{\pi}{2}}{1+\sin\frac{\pi}{2}}=0$ indeed.

Kinetic energy of a rotating body. Since rigid bodies have additional degrees of freedom compared to material points, energy must be spent on giving them angular speed. Let's consider a randomly-shaped rigid body rotating around some axis with angular speed ω. Its infinitesimally small part of mass m_i distanced by r_i from the axis has kinetic energy $\frac{m_i v_i^2}{2}=\frac{m_i(\omega r_i^2)}{2}$. To find the whole body's kinetic energy, we must sum up all its little pieces:

$$K_{rot}=\sum_i \frac{m_i(\omega r_i^2)}{2}=\frac{\omega^2}{2}\sum_i m_i r_i^2$$. But $\sum_i m_i r_i^2$ is called the moment of inertia of the body, so we can write

$$K_{rot}=\frac{I\omega^2}{2}$$, which fully corresponds to translational $K=\frac{mv^2}{2}$.

Problem 4. A ball (solid sphere) rolls down the incline of the height *h* without slipping. What speed will it have at the bottom of the incline?

Solution. If we considered the ball a material point with negligible moment of inertia, the conservation of energy $mgh=\frac{mv^2}{2}$ would immediately give us the answer $v=\sqrt{2gh}$. But here we must account for energy spent on the ball's rotation, so its final speed would be less. Now its kinetic energy includes not only the translational, but also rotational part:

$$mgh=\frac{mv^2}{2}+\frac{I\omega^2}{2} \quad .$$

From your textbook you can see (or derive it yourself) that the moment of inertia of a solid sphere about its diameter is $\frac{2}{5}mR^2$. Then, since the ball rolls without slipping, its angular speed is proportional to the speed of its center of mass: $\omega=\frac{v}{R}$. Substituting all this in our energy equation, we have

$$mgh=\frac{mv^2}{2}+\frac{1}{2}\frac{2}{5}mR^2\frac{v^2}{R^2}=\frac{mv^2}{2}+\frac{mv^2}{5}=mv^2\left(\frac{1}{2}+\frac{1}{5}\right)=\frac{7}{10}mv^2 \quad \text{, so} \quad v=\sqrt{\frac{10}{7}gh} \quad .$$

Angular momentum. In addition to energy and momentum, there is one more physical quantity which is conserved under certain conditions.

I will take the general ("advanced") definition of the torque, $\vec{M}=\vec{r}\times\vec{F}$ and set the goal to integrate it by time:

$$\int \vec{M}\,dt=\int \vec{r}\times\vec{F}\,dt \quad .$$

From second Newton's law, $\vec{F}=m\vec{a}=m\frac{d\vec{v}}{dt}=\frac{d}{dt}(m\vec{v})=\frac{d\vec{p}}{dt}$, so we can plug this in:

$$\int \vec{M}\,dt=\int \vec{r}\times\frac{d\vec{p}}{dt}dt \quad .$$

Integrating by parts, we have

$$\int \vec{M}\,dt=\int \vec{r}\times\frac{d\vec{p}}{dt}dt=\vec{r}\times\vec{p}-\int \vec{p}\times\frac{d\vec{r}}{dt}dt=\vec{r}\times\vec{p}-\int \vec{p}\times\vec{v}\,dt=\vec{r}\times\vec{p}-\int m\vec{v}\times\vec{v}\,dt=\vec{r}\times\vec{p} \quad ,$$

since the cross product of a vector with itself is always zero. In reverse,

$\frac{d}{dt}(\vec{r}\times\vec{p})=\vec{M}$, so when the torque on the rigid body is zero, the quantity $\vec{L}=\vec{r}\times\vec{p}=const$. This quantity is called *angular momentum*. It can be proven that, for a system of rigid bodies, angular momentum is conserved when the *external* torque equals zero (just like the linear momentum is conserved when the external force equals zero).

The magnitude of angular momentum of a material point is $L=|\vec{r}\times\vec{p}|=m|\vec{r}\times\vec{v}|=mvr\sin\alpha$, where α is the angle between the velocity vector and radius-vector. For a rigid body rotating about an axis with angular speed ω, it is the sum of all angular momenta of its pieces, $L=\sum_i m_i v_i r_i \sin\alpha$. Since $v_i=\omega r_i$ and for any piece the radius-vector from the axis is perpendicular to the speed, $\sin\alpha=\sin\frac{\pi}{2}=1$, and $L=\sum_i m_i\omega r_i^2=\omega\sum_i m_i r_i^2=I\omega$.

Problem 5. A small ball (which can be treated as a material point) is moving on the horizontal frictionless surface with speed *v* perpendicularly to the rod of the same mass and length ℓ which is resting on the surface (see Fig. 4), and hits the rod at one of its ends. Describe the motion of the system after the impact in two scenarios: 1) the ball sticks to the end of the rod and the two move together; 2) the ball elastically collides with the rod with no loss of energy.

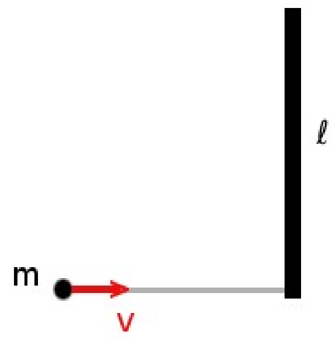

Fig. 4

Solution.
Scenario 1. After the impact the system (the rod with the ball stuck to its end) will be moving to the right and simultaneously rotating around its center of mass. Using the formula for the center of mass, we can find the center of mass of the system "rod+ball": put the axis x along to the rod and the origin at its end where the ball is. In this system, the ball's position (and center of mass) is $x = 0$ and the rod's center of mass is $x = \ell/2$, so the common center of mass is

$$x_c=\frac{m\cdot 0+m\cdot\frac{\ell}{2}}{m+m}=\frac{\ell/2}{2}=\frac{\ell}{4}$$

(the masses are equal). Thus, the center of mass is ¼ of the rod's length from the end with the ball.

Using conservation of momentum, we can find out the speed of the center of mass:
$mv=(m+m)v'$, $v'=\frac{v}{2}$. The system's center of mass will be moving with half the initial speed of the ball.

What about rotation? We have to consider rotation (and angular momentum) around the center of mass, which is ℓ/4 from the end with the ball. The initial angular momentum of the ball with respect to this point (which we shall call a pivot) is $L_1=mv\frac{\ell}{4}$. The final angular momentum of the whole system is $L_2=I\omega$, where *ω* is the angular speed of the system "rod+ball" around its center of mass.

We need to find *I* of this configuration about the pivot. The ball's *I* is $m(\frac{\ell}{4})^2$, since the ball is a small particle at the distance ℓ/4 from the pivot. The rod's *I* about its own center of mass is $\frac{m\ell^2}{12}$, and we can use Huygens – Steiner theorem to calculate the *I* of the rod about the pivot. Since $d=\frac{\ell}{4}$ in our case, the rod's I about the pivot would be $\frac{m\ell^2}{12}+md^2=\frac{m\ell^2}{12}+m(\frac{\ell}{4})^2$. Adding the ball's I to this, we get the I of the whole system:

$$I=\frac{m\ell^2}{12}+m(\frac{\ell}{4})^2+m(\frac{\ell}{4})^2=\frac{m\ell^2}{12}+2m(\frac{\ell}{4})^2=\frac{m\ell^2}{12}+\frac{m\ell^2}{8}=m\ell^2(\frac{1}{12}+\frac{1}{8})=\frac{5}{24}m\ell^2 \quad .$$

Returning to the angular momenta before and after the impact, we know that there were no external torques, so it was conserved: $L_1=L_2$, or

$$mv\frac{\ell}{4}=I\omega=\frac{5}{24}m\ell^2\omega \quad \text{, so} \quad \omega=\frac{6v}{5\ell} \quad .$$

<u>Answer:</u> the system's center of mass will be moving to the right with the speed $\frac{v}{2}$ and the system will be simultaneously rotating around it with angular speed $\frac{6v}{5\ell}$.

<u>Scenario 2.</u> In this case the ball will acquire a new speed and the rod will be rotating around its own center of mass (at $\ell/2$), while its center of mass will be moving with a certain speed. For translational motion, we can use the conservation of momentum:

$mv=mv_1+mv_2$, or $v=v_1+v_2$, where v_1 and v_2 are the speeds of the ball and of the center of the rod after the impact.

Energy is also conserved here, as mentioned in the problem. So, the initial kinetic energy of the ball will equal its final kinetic energy plus the total kinetic energy (translational and rotational) of the rod:

$\frac{mv^2}{2}=\frac{mv_1^2}{2}+\frac{mv_2^2}{2}+\frac{I\omega_2^2}{2}$, or $mv^2=mv_1^2+mv_2^2+I\omega_2^2$. The rod now rotates about its own center of mass, and in this case $I=\frac{m\ell^2}{12}$, so $mv^2=mv_1^2+mv_2^2+\frac{m\ell^2}{12}\omega_2^2$ or $v^2=v_1^2+v_2^2+\frac{\omega_2^2\ell^2}{12}$.

Finally, angular momentum is also conserved. With respect to the center of the rod, the ball's initial $L_1=mv\frac{\ell}{2}$ and the final angular momentum of both objects is $L_2=mv_1\frac{\ell}{2}+I\omega_2$. Setting them equal, we have $mv\frac{\ell}{2}=mv_1\frac{\ell}{2}+I\omega_2=mv_1\frac{\ell}{2}+\frac{m\ell^2}{12}\omega_2$, or $v=v_1+\frac{\omega_2\ell}{6}$.

Let us bring all three equations in a system:

$$\begin{cases} v=v_1+v_2 \\ v^2=v_1^2+v_2^2+\dfrac{\omega_2^2\ell^2}{12} \\ v=v_1+\dfrac{\omega_2\ell}{6} \end{cases} \quad .$$

We have three unknowns v_1, v_2, and ω_2. From the first and third equations we have $v_2 = \frac{\omega_2 \ell}{6}$, or $\omega_2 = \frac{6 v_2}{\ell}$. Substituting it into the second equation, we have

$$v^2 = v_1^2 + v_2^2 + \left(\frac{6 v_2}{\ell}\right)^2 \frac{\ell^2}{12} = v_1^2 + v_2^2 + \frac{36 v_2^2}{\ell^2} \frac{\ell^2}{12} = v_1^2 + v_2^2 + 3 v_2^2 = v_1^2 + 4 v_2^2 \quad .$$

Now, combining this result with the first equation, we can apply the trick we had in inelastic collision solutions in order not to mess with too much math:

$$\begin{cases} v = v_1 + v_2 \\ v^2 = v_1^2 + 4 v_2^2 \end{cases} , \quad \begin{cases} v - v_1 = v_2 \\ v^2 - v_1^2 = 4 v_2^2 \end{cases} , \quad \begin{cases} v - v_1 = v_2 \\ (v - v_1)(v + v_1) = 4 v_2^2 \end{cases} , \quad \frac{(v - v_1)(v + v_1)}{(v - v_1)} = \frac{4 v_2^2}{v_2} , \quad v + v_1 = 4 v_2 \quad .$$

Now the easy linear system $\begin{cases} v = v_1 + v_2 \\ v + v_1 = 4 v_2 \end{cases}$ gives us $v_1 = \frac{3}{5} v$, $v_2 = \frac{2}{5} v$. Therefore,

$$\omega_2 = \frac{6 v_2}{\ell} = \frac{6}{\ell} \frac{2}{5} v = \frac{12 v}{5 \ell} \quad .$$

<u>Answer:</u> after the impact, the ball will continue moving to the right with the speed $\frac{3}{5} v$, while the rod's center will be moving with the speed $\frac{2}{5} v$ and the rod will be rotating around it with angular speed $\frac{12 v}{5 \ell}$.