Work and Energy

Given the dynamical environment (forces), it is possible to solve for the object's motion (for example, find acceleration or speed) using Newton's laws. However, in some configurations it turns out to be much easier to use another method.

Let us take the familiar second Newton's law:

$$\sum_i \vec{F}_i = m\vec{a} \quad ,$$

where each $\vec{F}_i$ is the i^{th} force acting on the particle, *m* is its mass, and $\vec{a}$ is its acceleration.

Let's multiply it (using a dot product) by the particle's speed $\vec{v}$:

$$\sum_i \vec{F}_i \cdot \vec{v} = m\vec{a}\cdot\vec{v} \quad ,$$

and then integrate it with respect to time from one time moment to another:

$$\int_{t_1}^{t_2} \sum_i \vec{F}_i \cdot \vec{v}\, dt = m\int_{t_1}^{t_2} \vec{a}\cdot\vec{v}\, dt \quad ,$$

assuming that the mass of the particle stays constant and thus factoring it out of the integral.

Now notice that $\frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{v}^2) = \frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a} = 2\vec{a}\cdot\vec{v}$,

and therefore $\vec{a}\cdot\vec{v} = \frac{1}{2}\frac{d}{dt}(v^2)$, so the right-side integral in our equation will be

$\frac{1}{2}mv^2(t_2) - \frac{1}{2}mv^2(t_1)$, or just $\frac{mv_2^2}{2} - \frac{mv_1^2}{2}$, where at the times t_1 and t_2 we called the speeds v_1 and v_2 respectively.

The quantity $K \equiv \frac{mv^2}{2}$ is called the *kinetic energy* of the particle. Thus, we can say that the right-side integral equals the difference of kinetic energy at points 1 and 2.

But what about the left-side integral? As integration is linear, we can bring the integral under the sum: $\sum_i \int_{t_1}^{t_2} \vec{F}_i \cdot \vec{v}\, dt$, or $\sum_i \int_{t_1}^{t_2} \vec{F}_i \cdot d\vec{r}$, because $\vec{v}\, dt$ is obviously the infinitesimal displacement of the particle, $d\vec{r}$. The dot product $\vec{F}_i \cdot d\vec{r} \equiv dA$ is called *elementary work* of the force $\vec{F}_i$ on the displacement $d\vec{r}$, and the whole integral, which now does not have a time variable and is fully described by coordinates, is called work of the force $\vec{F}_i$ on the curve that the particle traveled along between times t_1 and t_2:

$$A_i = \int_C \vec{F}_i \cdot d\vec{r} \quad .$$

The integral is now a *line integral* along the curve C which begins at time t_1 and ends at time t_2.

Given all these definitions, let's return to the original equation. It follows that <u>the sum of all the works of each force (or the work of the resulting force) equals the change in kinetic energy</u>.

Given the line integral in the definition of work, calculating work is therefore not easy in most cases. Luckily, for some types of forces the work integral has an interesting property: the work does not depend on the curve path; therefore, only the initial point and the final point matter. Such forces are called *conservative* or *potential*, and for these the integral boils down to subtracting the two quantities related to the initial and final points. These quantities are called the *potential energy* of the particle at those points.

In this course only two types of potential are studied: gravitational potential and spring potential. These can be easily derived.

Gravitational potential. Let a particle fall free along a straight line. The only force acting on it is $m\mathbf{g}$. Since the force and the displacement are parallel, their dot product is just the product of their magnitudes, and since the force is constant, it goes outside of the integral, so the work is mg times the distance traveled:

$\int_C m\vec{g} \cdot d\vec{r} = \int_{z_1}^{z_2} m g\, dz = mg(z_2 - z_1) = mgh$, where z is the vertical coordinate and h is the height the particle lost. Thus, we can say that the particle has the potential energy $U = mgh$ where h is the height above the chosen zero level. You can choose your zero level anywhere you want because in the end only the difference of potential energies matters.

Spring potential. Displace a spring (for example, stretch) by a distance x. Hooke's law, which works for small spring displacements, says that the spring force is proportional to the displacement and opposite to it: $F = kx$, where k is called *stiffness* or *spring constant*. Therefore, to stretch a spring from equilibrium to some point x_0, you need to gradually increase the force with which you are pulling. At some point x in between, your force will be kx and the elementary work you are about to make by moving further by dx is $k\,x\,dx$. We can integrate all the elementary works to find the total work: $A = \int_0^{x_0} k\,x\,dx = \frac{k x_0^2}{2}$. Therefore, we can say that if you stretch (or compress) the spring by any distance x, it acquires the potential energy $U = \frac{k x^2}{2}$.

Let us return to our original equation:

$$\sum_i \int_C \vec{F}_i d\vec{r} = \frac{m v_2^2}{2} - \frac{m v_1^2}{2}$$

The right side can be written as $K_2 - K_1$. Now, some forces in the left side can have the above-mentioned properties (like gravity or spring forces), and therefore, their work is just the change in their respective potential energies (but with a minus sign)! So we can put them all on the right side together with the kinetic energies, and on the left we shall leave only the "bad" forces, for which one can't have an easy way (those are called *non-conservative*):

$\int_C \sum_i \vec{F}_i^{(nc)} \cdot d\vec{r} = K_2 - K_1 + U_2 - U_1$, or $A_{nc} = (K_2 + U_2) - (K_1 + U_1)$, where A_{nc} is the work of all non-conservative forces.

The sum of potential and kinetic energies is called the total mechanical energy E:

$$E_2 - E_1 = A_{nc} \quad .$$

Non-conservative forces in mechanics are dissipative forces like friction or air drag. Under ideal conditions, where none of them are present, the total mechanical energy is conserved:

$$E = const \quad \text{if} \quad A_{nc} = 0 \quad .$$

As an example, let's do a simple problem using Newton's laws and using energy.

Problem 1.

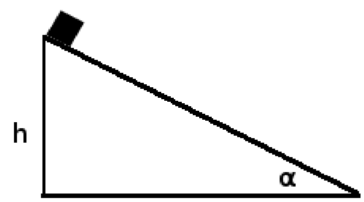

Fig. 1

In the setup above, find the final speed of the mass released from the top from rest. Neglect friction.

Solution (Newton). Identifying the two forces, **N** and *m***g**, and acceleration *a* along the incline, we project it on the axis parallel to the incline and get $ma = mg\sin\alpha, a = g\sin\alpha$. As seen from the triangle, the length of the slide is $l = \frac{h}{\sin\alpha}$, and from kinematics $v = \sqrt{2al}$, so

$v = \sqrt{2g\sin\alpha \frac{h}{\sin\alpha}} = \sqrt{2gh}$.

Solution (energy). Let us choose the zero potential for the gravity at the bottom of the incline. Therefore, on top of it the potential energy is $U_1 = mgh$ and kinetic energy K_1 is zero (because the mass is at rest). At the bottom, the potential energy $U_2 = 0$ (by our choice of the zero level) and the kinetic energy $K_2 = mv^2/2$ where v is the final speed we seek. As no friction is present, total mechanical energy is conserved:

$E_1 = E_2$, $U_1 + K_1 = U_2 + K_2$, $mgh + 0 = 0 + \frac{mv^2}{2}$, $mgh = \frac{mv^2}{2}$, and $v = \sqrt{2gh}$.

When solving the problem using energy, you don't have to take into account what happened between the initial and final points: you only compare the energies at the beginning and end. If there were no non-conservative forces in between, you need to write out the potential and kinetic energies at both states and set their sums equal.

Consider, for example, a problem similar to the one above and with the same question, but with a slightly different setup:

Fig. 2

Newton's equations and calculating the acceleration will be very difficult even if you know the equation of the curve in which the surface is shaped. If the curve is random, you can't do kinematics and forces here. But energy conservation will solve this problem! The initial and final potential and kinetic energies are the same here, so you will repeat the solution above and will get the same answer $v=\sqrt{2gh}$.

Problem 2. In the setup below, a mass released from rest from a height ℓ travels down onto the horizontal track which has a rough part (with friction) of the same length ℓ. Finally the mass hits a spring and stops when the spring is compressed the distance d. It is also known that if we attach the mass to this spring and let it hang, it will stretch the same distance d. Find the coefficient of friction of the rough part.

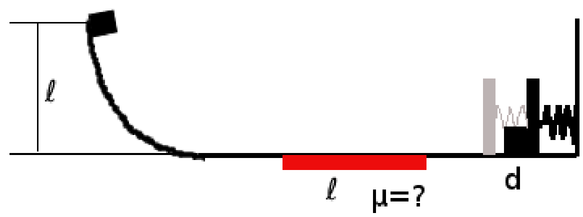

Fig. 3

Solution. The total mechanical energy at the starting point consists of zero kinetic (the mass is at rest) and the potential $mg\ell$ (the zero potential is chosen on the ground level). At the end point, the kinetic energy is zero, the gravitational potential energy is also zero (ground level), but the spring potential is $kd^2/2$, where k is the unknown stiffness of the spring. Energy is not conserved because there was a non-conservative force (friction) acting on the mass; its work is negative (friction is directed opposite to motion) and equals $-F_f\ell$; from Newton's equations we easily find $F_f = \mu mg$, so $A_f = -\mu mg\ell$.

$$E_2-E_1=A_{nc} \quad , \quad \frac{kd^2}{2}-mgl=-\mu mgl \quad , \quad \mu=1-\frac{kd^2}{2mgl} \quad .$$

Now we can use the piece of information saying that the spring stretches the same distance d when you hang the same mass on it. From equilibrium of the two forces, of the gravity mg and of the spring kd, we have $kd = mg$, so

$$\mu=1-\frac{kd\,d}{2mgl}=1-\frac{mgd}{2mgl}=1-\frac{d}{2l} \quad .$$