Vectors

Vectors are completely new mathematical objects with respect to numbers. In a given coordinate system, they are actually *sets of numbers* – in a 3-D Cartesian coordinate system, they are sets of three numbers each.

Let's visualize the difference between vectors and numbers by using an example. Suppose I have five oranges in my bag. To the question "How many oranges do you have in your bag?" the answer "Five" will be satisfactory. However, to the question "How far is your house from here?" the same answer "Five" will not be satisfactory: there must be something that has dimensions of length, say, km or miles. This is the important difference between pure numbers and physical quantities with dimensions that I pointed out on the Physics 1600 page.

Let's suppose now that I have to ask "But how can I come to your house?" and the answer is "Just walk or drive 5 km (or miles)". Again, the answer is not fully satisfactory: you need additional information, such as the direction. On the plane, two numbers are required to define the point (either the two coordinates with respect to the origin, or the length of the line to the point from the origin and the angle between this line and some preferred direction; in any case, you need two numbers), in space, three numbers. You can imagine a vector as an arrow that starts at one point and ends at another one, or as a set of three numbers which are the three displacements from the starting point with respect to the three possible directions that you need to make in order to arrive at the end point.

Vectors are denoted in writing by a variable with an arrow above it. For example, $\vec{a}.$ In textbooks they are often denoted by boldface, non-italic letters. For example, **a** (probably because it is easier for text editors). I will be using both here. The length, or magnitude, of a vector is denoted as $|\vec{a}|$ or as *a* – without the arrow.

Vectors are represented, in Cartesian coordinates, as sets of three numbers. For example,
$\vec{a}=\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ or $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$, where **i**, **j**, and **k** are vectors directed along the x, y, and z axes in the positive directions and whose length is 1. a_x, a_y, and a_z are the numbers called the *components* of the vector **a**. From here you can see that vectors can be **multiplied by any number**. What happens *geometrically* when you multiply a vector by a number is that its length gets multiplied by this number and its direction remains unchanged (if the number is positive) or changes it into the opposite (if the number is negative). In other words, the vector will stretch (multiplied by a number greater than one), compress (multiplied by a number between zero and one) and flip the direction (multiplied by a negative number).

Analytically, every component of a vector gets multiplied by this number. For example,

$$5\cdot\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}=\begin{pmatrix} 15 \\ -10 \\ 5 \end{pmatrix} .$$

Vectors can be **added** (and subtracted). Intuitively, you can guess that we will just add up each corresponding component. For example,

$\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}+\begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}=\begin{pmatrix} 1 \\ -3 \\ 10 \end{pmatrix}$. But what will happen geometrically? It turns out that there is an easy behavior of vector addition.

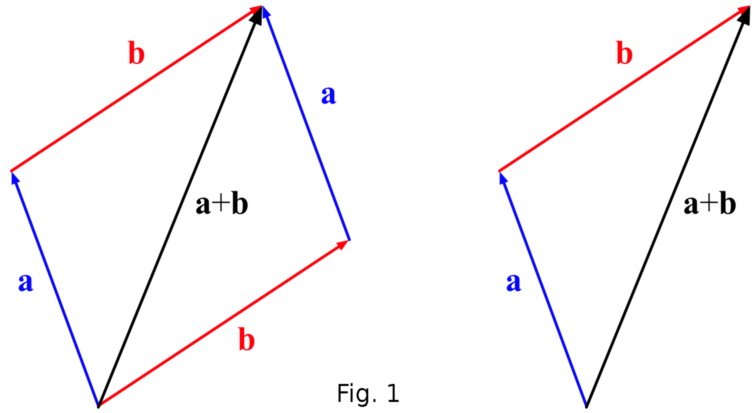

Fig. 1

In Fig. 1, you can see the addition of two arbitrary vectors **a** and **b**. You can use two ways:

1. Draw the parallelogram based on the vectors **a** and **b** drawn from the same point.. Its diagonal drawn from this point will be the sum of the two.
2. Draw the first vector and from the end of the first vector draw the second vector. The vector drawn from the beginning of the first vector to the end of the second vector will be the sum of the two.

The second rule can be generalized to any number of vectors. Draw each vector in the sum (for example, **a** + **b** + **c** + ...) from the beginning of the previous one, and the vector drawn from the beginning of the first vector to the end of the last one will be the resulting vector.

Subtraction of vectors is, of course, equivalent to addition of the opposite vector times -1, i.e., addition of the opposite vector: **a** – **b** = **a** + (–**b**). In the picture above, if we call the resulting vector **a** + **b** ≡ **c**, then, logically, **c** – **a** should equal **b**. From the picture you can see that **c** and **a** have the common beginning, and **b** is drawn from the end of **a** to the end of **c**. Thus, we can formulate the subtraction rule: to subtract two vectors, draw them from a common point and draw the vector between their tips which begins at the tip of the one which you are subtracting. In the parallelogram, whose diagonal, according to rule 1 above, is the sum of the vectors, you can see that the other diagonal is their difference.

Much more than all this, you will have to do the **projection** of vectors. In most physics problems you will need to project vectors on axes in order to formulate and continue calculations.

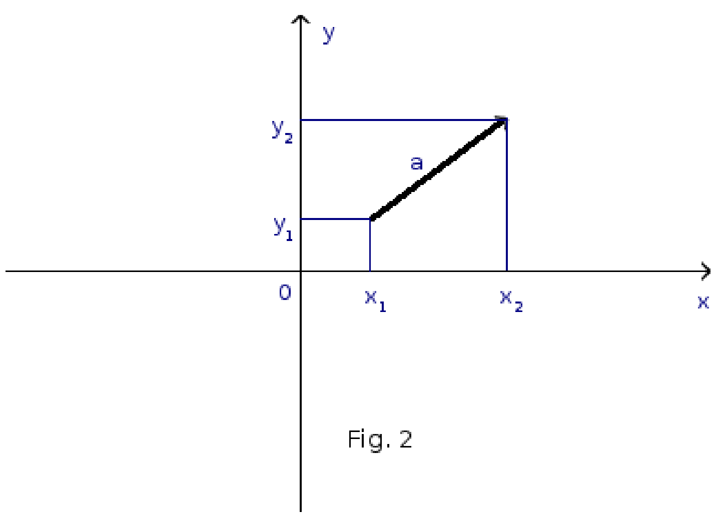

Fig. 2

A projection of a *point* on the axis is the intersection of the perpendicular drawn from this point onto the axis, with this axis. In Fig. 2, x_1 is the projection on the x-axis (or x-projection) of the beginning of vector **a**, and y_2 is the y-projection of the end of vector **a**. The projection of a *vector* will therefore be the difference of the projections of the end and beginning. Thus, the x-projection of **a** is $x_2 - x_1$, and its y-projection is $y_2 - y_1$.

You can notice that the projections are nothing but the good old components of the vector. Indeed, $(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} = \vec{a}$. Now we can use the terms "projection" and "component" interchangeably.

In most problems you will know the magnitude of a vector and the angle with a certain axis, and you will need to project it. For example, consider the situation below.

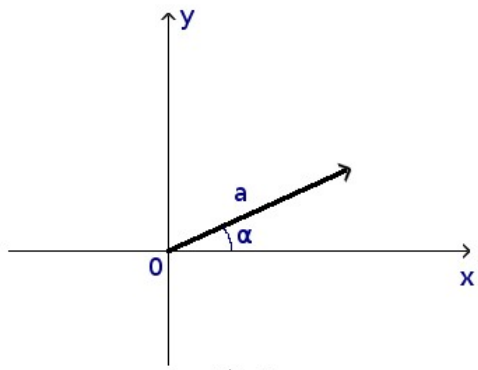

Fig. 3

We know that the magnitude of the vector **a** is *a* and the angle between it and the x-axis is *α*. By building perpendiculars from the end of the vector to the axes and considering the triangles, we see that the x-projection is *a* cos *α* and the y-projection is *a* sin *α*. Useful things to remember: if the vector projects on one axis with sine, it will project on the other (perpendicular) axis with cosine. Also, a mnemonic rule I made for English speakers is that if by projecting the vector it looks like you are "closing" the angle by moving the vector towards the desired axis, it will be the cosine of that angle (**c**lose – **c**osine).

Example problem 1. In Fig. 4 below (not to scale), *m*, *g*, and *μ* are known quantities, α is the angle with the horizontal, the vector **N** is perpendicular to *μ***N**, and *m***g** is straight down. Knowing that all the vectors sum up to zero, find *N* and *α*.

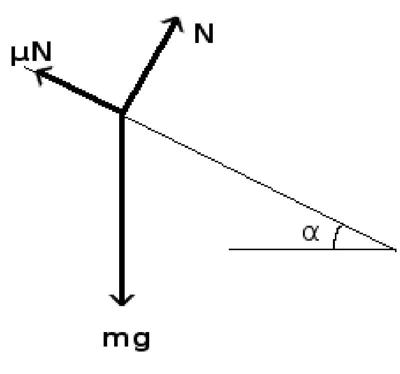

Fig.4

Solution. Since they all sum up to zero, their x- and y-projections also should sum up to zero (otherwise their result would have a non-zero component and would not be zero). Let us choose the x and y axes as usual, x horizontally to the right and y vertically upward, and do the projections.

x-projections: it can be seen that **N** has the angle α with the vertical, so its x-projection is $N \sin \alpha$. Then, μ**N** is under the angle α with the horizontal but directed to the left (our negative x-direction), so its x-projection is $-\mu N \cos \alpha$. m**g** is perpendicular to the x-axis, so its beginning and end project into the same point, making its projection zero. Based on the fact that all three should sum up to zero, we get the first equation:

$$N \sin \alpha - \mu N \cos \alpha + 0 = 0$$

y-projections: as I mentioned earlier, it's easy to remember that those vectors which had a sine in their x-projections will have a cosine here, and vice versa (but watch the direction too!). Here, **N** projects into $N \cos \alpha$, μ**N** into $\mu N \sin \alpha$, and the projection of m**g**, which is straight down, will be $-mg$. We have the second equation:

$$N \cos \alpha + \mu N \sin \alpha - mg = 0$$

This is a system with two equations and two unknowns: N and α. From the first one, we immediately find α:

$$N \sin \alpha = \mu N \cos \alpha, \sin \alpha = \mu \cos \alpha, \mu = \tan \alpha, \alpha = \arctan \mu .$$

Using the second equation, we find N:

$$N \cos \alpha + \mu N \sin \alpha = mg, N(\cos \alpha + \mu \sin \alpha) = mg, N = \frac{mg}{\cos \alpha + \mu \sin \alpha} = \frac{mg}{\cos \alpha (1 + \mu \tan \alpha)} ,$$

and, using the trigonometrical identity $\frac{1}{\cos^2 \alpha} = \tan^2 \alpha + 1$ and the fact that $\tan \alpha = \mu$, we get:

$$N = \frac{mg}{\cos \alpha (1 + \mu \tan \alpha)} = \frac{mg \sqrt{\tan^2 \alpha + 1}}{1 + \mu \tan \alpha} = \frac{mg \sqrt{1 + \mu^2}}{1 + \mu^2} = \frac{mg}{\sqrt{1 + \mu^2}} .$$

So, $\alpha = \arctan \mu, N = \frac{mg}{\sqrt{1 + \mu^2}}$ is the answer.

Dot product and cross product. Multiplication by a number and addition-subtraction are not the only operations you could do with vectors. These two will be also necessary for you to know. You won't need these for two-dimensional motion and Newton's laws, but will need them later in the course.

Both operations have the analytical definition (in terms of the vectors' components) and the geometrical one (in terms of the vectors' lengths and positions with respect to each other). It is convenient to use either, depending on whether you have the components or the magnitudes. Sometimes you have to combine both to get the results (see the example problem further).

Dot product (also known as scalar product) is an operation on two vectors whose output is a scalar, a single number (**not** a vector). It is defined as follows.

Analytical definition: Given two vectors $\vec{a}=\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}$, their dot product is

$$\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z \quad .$$

Geometrical definition: Given two vectors $\vec{a}$ and $\vec{b}$ with magnitudes *a* and *b* and the angle *α* between them, their dot product is

$$\vec{a}\cdot\vec{b}=ab\cos\alpha \quad .$$

You can see that the dot product is commutative (**a · b** = **b · a**). Also, from the geometrical definition it is clear that if the two vectors are perpendicular ($\alpha = 90°$), then their dot product is zero, and when they are parallel ($\alpha = 0°$ or $\alpha = 180°$), their dot product is the product of their magnitudes (positive for 0° and negative for 180°). Particularly, if **a** = **b**, **a · a** ≡ $\mathbf{a}^2 = a^2$, i.e., the vector dotted into itself equals its magnitude squared.

Example problem 2. For two arbitrary vectors $\vec{a}=\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}$, find the angle between them in terms of their components. Do a numeric calculation for $\vec{a}=\begin{pmatrix}4\\-1\\3\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}2\\-2\\4\end{pmatrix}$.

Solution. From the analytical definition, $\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z$. From the numerical definition, $\vec{a}\cdot\vec{b}=ab\cos\alpha$. Therefore, $a_x b_x+a_y b_y+a_z b_z=ab\cos\alpha$. The magnitudes of each vector in terms of its components can be found from using the Pythagorean theorem twice on each vector:

$$a=\sqrt{a_x^2+a_y^2+a_z^2},\ b=\sqrt{b_x^2+b_y^2+b_z^2}$$. Therefore,

$$a_x b_x+a_y b_y+a_z b_z=\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}\cos\alpha$$, and

$$\alpha=\arccos\frac{a_x b_x+a_y b_y+a_z b_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}} \quad .$$

For the numeric values given, $\alpha=\arccos\dfrac{4\cdot2+(-1)\cdot(-2)+3\cdot4}{\sqrt{4^2+(-1)^2+3^2}\sqrt{2^2+(-2)^2+4^2}}=\arccos 0.881=28.2\,°$.

Cross product (also known as vector product) is an operation on two vectors whose output is **another vector**. It is defined as follows.

Analytical definition: Given two vectors $\vec{a}=\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}$, their cross product is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \quad .$$

<u>Geometrical definition</u>: Given two vectors $\vec{a}$ and $\vec{b}$ with magnitudes *a* and *b* and the angle *α* between them, their cross product has the *magnitude* $|\vec{a}\times\vec{b}|=ab\sin\alpha$ and its *direction* is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ and directed such that $\vec{a}$, $\vec{b}$, and $\vec{a}\times\vec{b}$ form a right-hand triple (see Fig. 5 below).

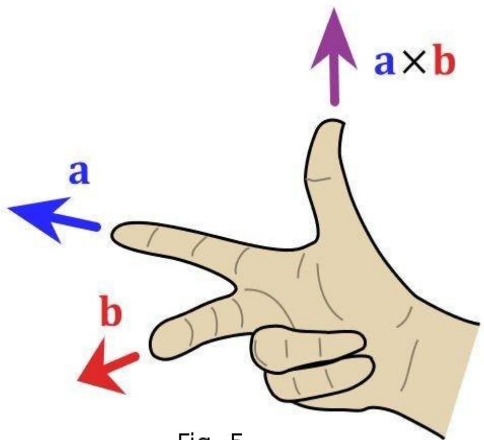

Fig. 5

Some interesting facts follow from these definitions. First, **a × b** is *non-commutative* – in fact, it is anti-commutative: **a × b = – b × a**. Second, you can't do the cross product in two dimensions – the resulting vector must be perpendicular to the plane of the two, and thus needs the third dimension. Third, from the geometrical definition it follows that the cross product of two parallel vectors is zero – sin 0º = sin 180º = 0.

The analytical definition uses the determinant. If you are unfamiliar with it, see http://en.wikipedia.org/wiki/Determinant and the reference links, or use any linear algebra book. Particularly, our determinant in the definition expands in the following way:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\vec{i}\,(a_y b_z-a_z b_y)-\vec{j}\,(a_x b_z-a_z b_x)+\vec{k}\,(a_x b_y-a_y b_x)=\begin{pmatrix}a_y b_z-a_z b_y\\ a_z b_x-a_x b_z\\ a_x b_y-a_y b_x\end{pmatrix} \quad .$$

Example problem 3. Find the cross product of $\begin{pmatrix}3\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}4\\2\\2\end{pmatrix}$.

Solution. $\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -1 & 1\\ 4 & 2 & 2\end{vmatrix}=\vec{i}\,(-2-2)-\vec{j}\,(6-4)+\vec{k}\,(6-(-4))=\begin{pmatrix}-4\\-2\\10\end{pmatrix}$.